Nonlinear and linear network theory

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1. **Lyapunov function for a symmetric network**

   (a) Show that the function

   $$L(x_1,\ldots,x_N) = \sum_i \bar{F}(x_i) - \sum_i b_i x_i - \frac{1}{2}\sum_{ij} W_{ij} x_i x_j \tag{1}$$

   is nonincreasing under the dynamics

   $$\frac{dx_i}{dt} + x_i = f\left(b_i + \sum_j W_{ij} x_j\right) \tag{2}$$

   assuming that $W_{ij} = W_{ji}$, f is monotonic, and $\bar{F}' = f^{-1}$. Why does the proof break down if W is not symmetric? (Hint: use the chain rule to show that $dL/dt \leq 0$).

   (b) Show that $dL/dt = 0$ only at steady states of the dynamics.

   (c) Consider the special case of $f(x) = \tanh(x)$. Show directly from the dynamics (2) that if the x_i are all initialized in the range $[-1, 1]$, then they must all stay within that range for all time.

   (d) What is $\bar{F}(x)$ for this case? Show that L is lower bounded in the domain $[-1, 1]^N$. Combined with the fact that L is everywhere decreasing, except at steady states where it is stationary, it follows that L is a Lyapunov function. (The condition of radial unboundedness is not necessary because the domain is bounded).

2. **The flip-flop**

   (a) Simulate the flip-flop dynamics

   $$\dot{x}_1 + x_1 = f(-\beta x_2) \tag{3}$$
   $$\dot{x}_2 + x_2 = f(-\beta x_1) \tag{4}$$

   where $f(x) = \tanh(x)$ using XPP, MATLAB, or whatever software you choose. Experiment with different values of β and different initial conditions. How many steady states are there for small β? How many are there for large β? Which are stable?

   (b) Perform a linear stability analysis of the symmetric steady state $x_1 = x_2 = 0$. Find the critical value of β at which multistability emerges.

   (c) Write down a Lyapunov function for the dynamics using the general result given in the previous problem. Try to visualize the function for different values of β.

3. **Lateral excitation and inhibition in a linear network.** Consider the N-neuron linear network

   $$\tau\frac{dx_i}{dt} + x_i = b_i + \sum_{j=1}^{N} W_{ij} x_j \tag{5}$$

where W is a *circulant* matrix

$$W = \begin{pmatrix} w_0 & w_{N-1} & . & . & w_1 \\ w_1 & w_0 & w_{N-1} & . & w_2 \\ . & w_1 & w_0 & . & . \\ . & . & . & . & w_{N-1} \\ w_{N-1} & w_{N-2} & . & w_1 & w_0 \end{pmatrix} \tag{6}$$

This form means that the interaction strength between two neurons depends only on their separation, if the neurons are placed on a ring. The vector b consists of N inputs to the neurons, and the vector x consists of N neural activities.

In matrix-vector form, the fixed point equations are

$$x = b + Wx \tag{7}$$

(a) Show that the Fourier modes are eigenvectors of W, and find the eigenvalues of these modes in terms of the entries in the circulant matrix. Make sure that you find all N Fourier modes (there is a finite number because of periodic boundary conditions on the ring).

(b) Consider the case where $w_1 = w_{N-1} = c$ and the rest of the entries are zero. For what values of c is the network stable? In the following questions, restrict c to these values.

(c) Specialize to the inhibitory case $c < 0$.

i. Solve for the steady-state $x = (I - W)^{-1}b$ when there is a single input on, and all the rest are zero (impulse response). Describe verbally and sketch graphically the qualitative features of the solution.

ii. Sketch the eigenvalue spectrum of the gain matrix $(I - W)^{-1}$. Which modes are amplified and which are attenuated?

(d) Specialize to the excitatory case $c > 0$.

i. Solve for the steady-state $x = (I - W)^{-1}b$ when there is a single input on, and all the rest are zero (impulse response). Describe verbally and sketch graphically the qualitative features of the solution.

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4. **Linear network integrator** Consider the same linear network as in Eq. (5), but this time with weight matrix diagonalized as

$$W_{ij} = \sum_\mu \tilde{W}_\mu \xi_i^\mu \eta_j^\mu \tag{8}$$

where the right (ξ) and left eigenvectors (η) satisfy the orthogonality conditions

$$\delta_{ij} = \sum_\mu \xi_i^\mu \eta_j^\mu \tag{9}$$

$$\delta_{\mu\nu} = \sum_i \xi_i^\mu \eta_i^\nu \tag{10}$$

(a) **Conditions for a line attractor** Show that the conditions for a line attractor are that

- A single eigenvalue must be tuned to exactly one.
- The real parts of the other eigenvalues must be less than one.
- The left eigenvector of the unity eigenvalue must be orthogonal to b.

(b) **Conditions for the VOR** Suppose that b contains vestibular input, so that it is modulated by head velocity $\dot{H}$

$$b_i = b_i^0 + \dot{H}\zeta_i \tag{11}$$

And let eye position be read out by the oculomotor plant according to

$$\tau_E \dot{E} + E = \sum_i \rho_i x_i \tag{12}$$

Analyze the linear transformation from H to E. What are the conditions that the system must satisfy to have a good VOR ($\dot{E} = -\dot{H}$)? Hint: in general, this can be satisfied only approximately. It might be good to start with the rank one case ($W_{ij} = \xi_i \eta_j$) and then generalize.