A single nonlinear feedback loop

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1 Graphical solution

Let’s analyze the dynamics $\dot{x} + x = f(b + Wx)$. Before we considered the linear case $f(x) = x$, but now consider a sigmoidal nonlinearity $f(x) = \tanh x$. This is a convenient choice, because it behaves approximately linearly near $x = 0$, but saturates for large $x$.

Suppose that the input $b$ is constant in time, and find the steady states of the dynamics by solving $x = f(b + Wx)$. This can be done graphically by plotting $y = x$ and $y = f(b + Wx)$, and looking for their intersection points. Equivalently, $f^{-1}(x) = b + Wx$ can be solved by looking at the intersection of $y = f^{-1}(x)$ and $y = b + Wx$.

2 Weak positive feedback

If $W$ is small and positive, there is only one intersection point for any input $b$. This is like the linear case, since there is a unique steady state response for any input.

If $b = 0$, then the intersection is at $x = 0$. If $b$ is small but nonzero, then $x$ is roughly linear in $b$. But if $b$ becomes large, the response of $x$ saturates. So the system acts as a nonlinear amplifier.

The relationship between $x$ and $b$ is nonlinear, so we can’t describe it by a single gain. Instead, gain is a function of $b$, and can be defined in various ways. The differential gain is defined as $dx/db$. Since $dx = f'(b + Wx)(db + Wdx)$, it follows that

$$\frac{dx}{db} = \frac{1}{f'(b + Wx)^{-1} - W}$$

Note that in the linear case, this reduces to $dx/db = 1/(1 - W)$. This formula is approximately valid for the nonlinear case $f(x) = \tanh x$ when the argument $b + Wx$ is small. But when the argument is large, the differential gain approaches zero, because the response is insensitive to changes in input when the system is operating in saturation.
3 Strong positive feedback

If $W$ is large, there can be more than one intersection point, depending on the input $b$. This is very unlike a linear system, as more than one steady state response is possible for a single input.

- If $b = 0$, then there is one steady state at $x = 0$. However, there are also two nonzero steady states, one positive and one negative. It can be shown that the steady state at $x = 0$ is unstable, while the two other steady states are stable. Thus the system is bistable. In general, a system with more than one stable state is called multistable.
- For large $W$ it is possible to derive an approximate expression for the stable states, which are close to 1 and $-1$.
- If $b$ increases in magnitude, or $W$ decreases, eventually the bistability will disappear.

4 Bifurcation analysis

- Let’s vary $W$ with $b = 0$ and do a bifurcation analysis. The bifurcation point occurs when $y = x$ is tangent to $y = f(Wx)$ at the steady state $x = 0$. This means that $f'(0)W = 1$, which implies that $W = 1$ for $f(x) = \tanh x$.
- Let’s vary $b$ with $W > 1$ fixed. At the extremes of the graph, there is only one steady state, and it is stable. In the middle of the graph, there are three steady states, two of which are stable. Due to this bistability, sweeping the input from low to high and back to low again results in a hysteresis loop.

5 Energy function

We can write the dynamics as

\[
\frac{dx}{dt} = -\frac{dE}{dx}
\]

where

\[
E = \frac{1}{2}x^2 - \frac{1}{W} F(b + Wx)
\]

where $F' \equiv f$. In the special case of $f(x) = \tanh x$, $F(x) = \ln \cosh x$.

6 Negative feedback

In this case, there is always only one steady state of the dynamics, and it is stable.