Problem Set 4 (due Thurs Mar. 6)
Fourier series and integrals

Feb. 27, 2003

1. **Fourier series for a sawtooth** Define the sawtooth function by $s(t) = t/\pi$ for $-\pi < t < \pi$.

   (a) In the Fourier series for the sawtooth,

   $$s(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

   what values does ω_n take on? Find the coefficients a_0, a_n, and b_n by computing the integrals

   $$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} dt\, s(t) \quad (1)$$

   $$a_n = \frac{2}{T} \int_{-T/2}^{T/2} dt\, s(t) \cos \omega_n t \quad (2)$$

   $$b_n = \frac{2}{T} \int_{-T/2}^{T/2} dt\, s(t) \sin \omega_n t \quad (3)$$

   $$(4)$$

   where $T = 2\pi$ is the length of the interval $(-\pi, \pi)$.

   (b) Write MATLAB code to sum the first 50 terms of the Fourier series, and graph the result. This way you can check whether you did the integrals right.

2. **The Fourier modes as an orthonormal basis** Define the inner product of two functions f and g on the interval $[-T/2, T/2]$ as

   $$\langle f, g \rangle = \frac{2}{T} \int_{-T/2}^{T/2} dt\, f(t) g(t)$$

   Show that the Fourier modes $1/\sqrt{2}$, $\cos \omega_n t$, and $\sin \omega_n t$ for $\omega_n = 2\pi n/T$ constitute an orthonormal set of functions, where n ranges over all integers from $-\infty$ to ∞. In other words, show that the inner product of each function with itself is unity, and that the inner product of each function with a different function vanishes.

   This is why the Fourier modes can be regarded as a set of perpendicular axes in the infinite-dimensional space of functions on the interval $[-T/2, T/2]$. The formulas for the Fourier coefficients are just projections onto the Fourier modes (projection is just the term for inner product).

3. **Chirp** Generate a five second long chirp at a frequency that decays exponentially from an initial value of 1000 Hz with a time constant of $\tau = 5$ seconds,

   $$\nu(t) = 1000 e^{-t/\tau}$$

   The chirp itself is given by

   $$x(t) = \cos(2\pi \nu(t) t)$$

Use a sampling frequency of 22050 Hz. Listen to the chirp for fun. Type `specgram(x,8192,22050)` to display the spectrogram. If you generated the chirp right, you should see the exponential decay of the frequency starting from 1000 Hz.

4. **Nonlinear distortion**. With MATLAB, generate a signal composed of two pure tones with frequencies $\nu_1 = 1000$ Hz and $\nu_2 = 1100$ Hz.

$$x(t) = \cos(2\pi\nu_1 t) + \cos(2\pi\nu_2 t)$$

Use a sampling frequency of 22050.

(a) Run the command `spectrum(x,[],[],[],22050)`. Why is there only a single peak?

(b) Play around with the parameters of the `spectrum` command until you find a way to make the peaks at 1000 Hz and 1100 Hz both visible.

(c) Now display the spectrum of the square of the signal, x^2. Use the same parameters in the spectrum command so that all the peaks are visible. You should see that x^2 contains frequencies that were not present in the original signal. This is a mark of a nonlinear system, and is called harmonic and intermodulation distortion by audiophiles. They avoid nonlinearity like the plague.

(d) Find a formula that expresses x^2 as a sum of sinusoids. You can think about it as a Fourier series, or just use trig identities. Use your formula to explain the locations of the peaks in the spectrum of x^2.

5. **Fourier transform of a Gaussian**

(a) Show that

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi} \tag{5}$$

Hint: The square of the integral can be written as a double integral

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)/2}$$

which can be evaluated upon transformation to polar coordinates.

(b) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2+ax} = e^{a^2/2} \tag{6}$$

You may assume that a is a real number in your derivation. Hint: Complete the square in the integrand.

(c) In one dimension, the Gaussian probability density is defined as

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

This is centered at μ, and has width σ. Note that Eq. (5) implies that $\int_{-\infty}^{\infty} dx\, P(x) = 1$.
Calculate the Fourier transform of $P(x)$,

$$\hat{P}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} P(x)$$

Hint: Assume that Eq. (6) is valid for complex a.
Plot $\hat{P}(k)$ and $P(x)$ on the same graph for $\mu = 0$ and $\sigma = 2$. Which function is wider? Which function is taller?

6. **The first-order lowpass filter**

(a) In the second problem set, we encountered the first-order lowpass filter in the form of the differential equation

$$\tau \frac{dy}{dt} + y = x$$

Now let's treat it with Fourier methods. Before we proved that the impulse response is

$$h(t) = \tau^{-1} \exp(-t/\tau)\Theta(t)$$

where $\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ for $t < 0$. We also saw that y is equal to the convolution of h with x.

Show that the Fourier transform of the impulse response is

$$\hat{h}(\omega) \equiv \int_{-\infty}^{\infty} dt\, e^{-i\omega t} h(t) \quad (7)$$

$$= (1 + i\omega\tau)^{-1} \quad (8)$$

Hint: You know that

$$\int_{0}^{\infty} dt\, e^{-at} = \frac{1}{a}$$

holds for real a. Assume that this is also true for complex a,

(b) Calculate the magnitude of the complex amplitude $\hat{h}(\omega)$, and sketch it as a function of ω. Does it increase or decrease with ω? What happens to the amplitude of a sinusoid that is put through the low-pass filter? Based on your answer, can you explain why the filter is called "lowpass"?

(c) Calculate the phase of the complex amplitude $\hat{h}(\omega)$, and sketch it as a function of ω. Does it increase or decrease with ω? What happens to the phase of a sinusoid that is put through the low-pass filter?